Representation Stability, Configuration Spaces, and Deligne–Mumford Compactifications

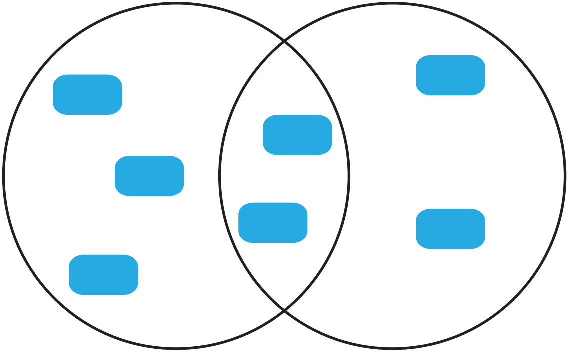
Phil Tosteson

Thesis Defense, April 16 2019

Intro to Topology

Inclusion–Exclusion

How many blue blobs are there?



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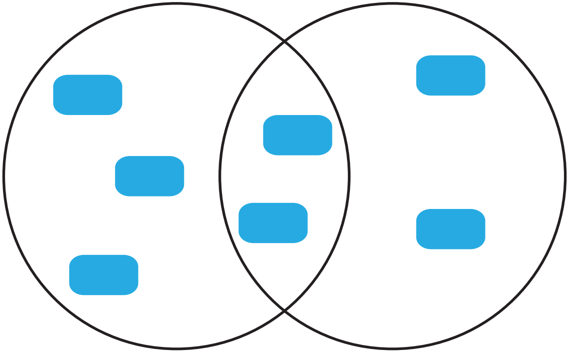
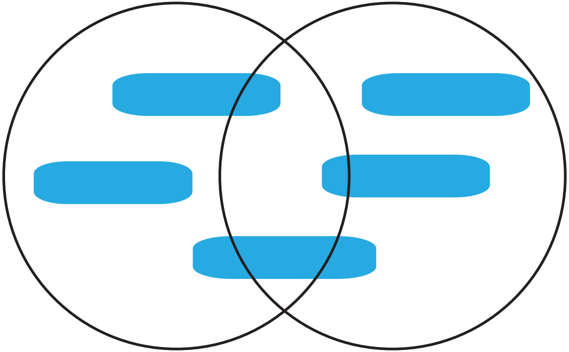


There are $5 + 4 - 2$, so 7 in total

With Overlaps

Does inclusion exclusion still work when there are overlaps?



We have $4 + 4 - 3$

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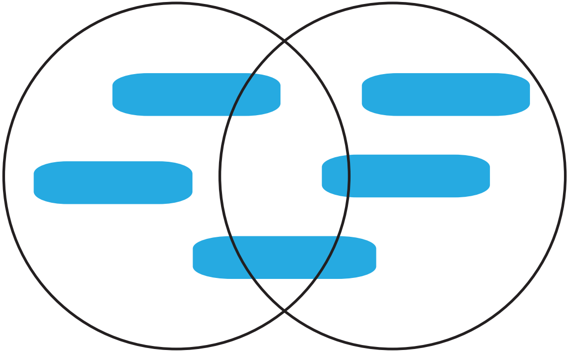
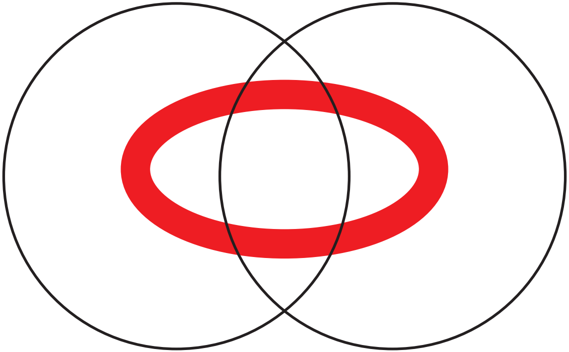


We have $4 + 4 - 3 = 5$, so yes!

Or does it?

We have $1 + 1 - 2 = 0$. What is going on?



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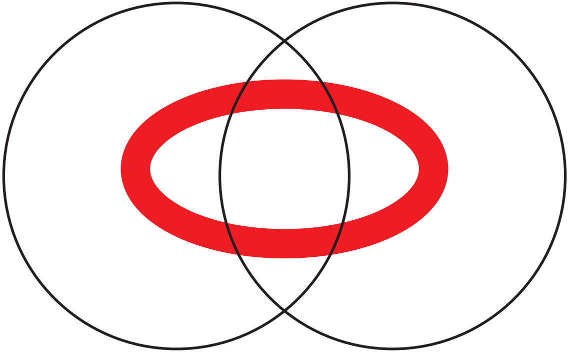

To a topologist, this makes sense because 0 is the **Euler Characteristic** of the annulus.

Euler Characteristic

The **Euler Characteristic** was the first ever topological invariant. It is a topological version of counting.

- It is a number $\chi(A)$, assigned to any shape (topological space) A
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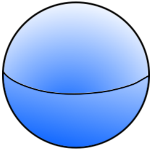
- You can compute it by breaking A up into pieces and using inclusion exclusion.
- **No matter how** you break up A, you get the same answer!
- It only depends on the topology of A, not on how "big" or "sharp" A is.

Euler Characteristic of the 2 sphere

This is the 2-sphere, S^2



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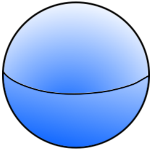
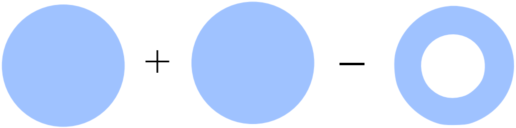


To compute $\chi(S^2)$, we break it up into the top piece and the bottom piece.



So we get $\chi(S^2) = 1 + 1 - 0 = 2$.

Homology

The homology of a space X is a **refinement** of its Euler characteristic, invented by Poincaré

- There is a vector space $H_i(X)$ for every $i \in \mathbb{N}$.
- Recording dimensions gives us a sequence of **betti numbers** $\{\dim H_i(X)\}_{i\in\mathbb{N}}$.
- We have
$$\chi(X) = \sum_i (-1)^i \dim H_i(X)$$
- This talk will be about computing the vector spaces $H_i(X)$, for certain topological spaces.

Configuration Spaces

Configuration Space

- We let X be a Hausdorff topological space
- The **ordered configuration space** of n points in X is

$$\mathrm{Conf}_n(X) = \{(x_i) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

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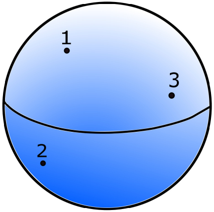
- We visualize configuration space as n labelled points moving around in X, where the points are not allowed to collide.

Configuration Space of a Sphere

This picture represents an a point in the space $\mathrm{Conf}_3(S^2)$.



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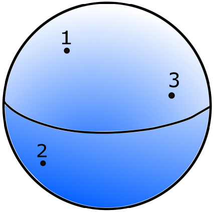

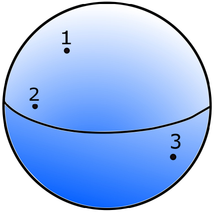

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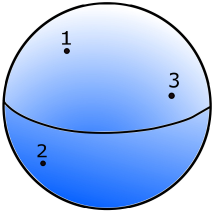

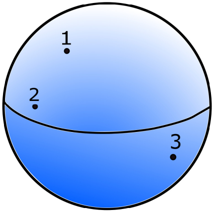

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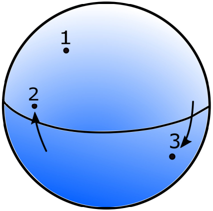
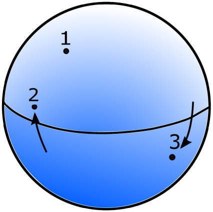


Here is a path from the first point to the second point.

Configuration Space of a Sphere



Since there are 3 points we can vary, and 2 degrees of freedom for each point, the space $\mathrm{Conf}_3(S^2)$ is $2 \cdot 3 = 6$ dimensional.

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- An embedding $X \hookrightarrow Y$ induces a map on configuration spaces $\mathrm{Conf}_n X \to \mathrm{Conf}_n Y$. Can use this to study **knots**! $S^1 \hookrightarrow \mathbb{R}^3$. (In general, we have embedding calculus).

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- Topological complexity and motion planning.

Question:

What is the (co)homology of $\mathrm{Conf}_n(X)$? As an $\mathbf{S_n}$ representation?

- Answer should involve the cohomology of X
- But $X \mapsto \mathrm{Conf}_n(X)$ does not preserve homotopy equivalences– this suggests we need more!
- When $n >> 0$, we hope that $H^i(\mathrm{Conf}_n(X))$ admits a uniform description.

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For example, this is the category **FI**, of finite sets and injections.

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An **FI** module is a functor from **FI** to the category of abelian groups.

$$M_0 \to M_1 \rightrightarrows M_2 \overset{6}{\rightrightarrows} M_3 \overset{24}{\rightrightarrows} M_4 \overset{120}{\rightrightarrows} M_5 \rightrightarrows \cdots$$

Each M_n is an $\mathbf{S}_n$ representation. These representations are related by transition maps.

Finitely generated **FI** modules

An **FI** module is **finitely generated** if there is a finite list of elements $x_i \in M_{n_i}$, such that any $m \in M_n$ has the form

$$m = \sum_i a_i f_i x_i,$$

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- Integral / char p work of Nagpal and others.

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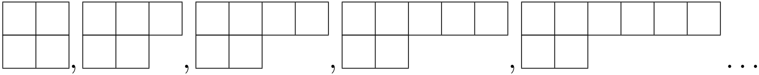

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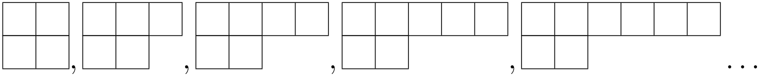


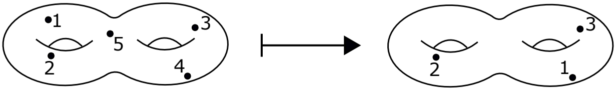
- We have that the symmetric function corresponding to M is a finite sum of the form

$$\sum_{\lambda} \pm s_\lambda \cdot \left(\sum_{n=0}^{\infty} h_n \right)$$

FI action on configuration space

An injection $f : [n] \hookrightarrow [m]$ gives a map $\mathrm{Conf}_n X \leftarrow \mathrm{Conf}_m X$, by **forgetting and relabelling points.**

For example, the injection $[3] \to [5]$, given by $1 \mapsto 4, 2 \mapsto 2, 3 \mapsto 3$ acts by



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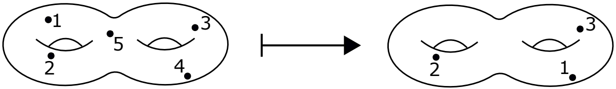

This means that $H^i(\mathrm{Conf}_n(X))$ is an **FI** module.

Representation stability for manifolds

Building on the work of Church, Church–Ellenberg–Farb showed

Theorem (CEF)

Let M be a connected manifold of dimension ≥ 2. Then for every i, $H^i(\mathrm{Conf}_n M, \mathbb{Q})$ is a finitely generated **FI** *module.*

This fails for $\mathbb{R}^1$!

Representation Stability for Non-Manifolds

We say that a point $p \in X$ is a **roadblock** if for $U \ni p$ a contractible neighborhood, $U - p$ is disconnected.

- Every point of a graph G is a roadblock
- $G \times \mathbb{R}^1$ has no roadblock points
- A wedge of two spaces has a roadblock point at the wedge
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Theorem (T.)

Let X be a finite connected CW complex or algebraic variety. Let k be a field. If X has **no roadblocks** *then*

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Extends to reasonable locally contractible closed subsets of $\mathbb{R}^n$.

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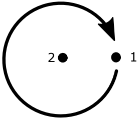
What are these methods?

Homology of $\mathrm{Conf}_n\mathbb{R}^2$

We can represent homology classes in $\mathrm{Conf}_n\mathbb{R}^2$ graphically.



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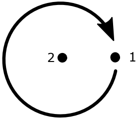

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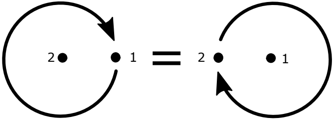

These two classes are equal

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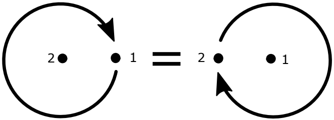
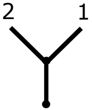
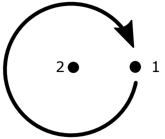


We can represent the class by this tree



Homology of $\mathrm{Conf}_n\mathbb{R}^2$

How does $\mathbf{S}_2$ act on this class?



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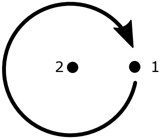

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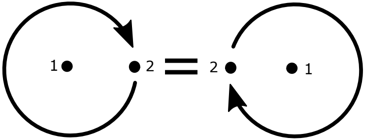

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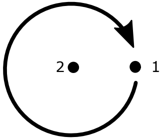

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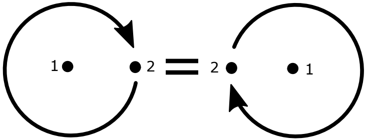
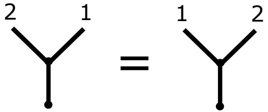
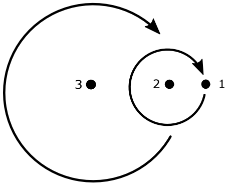


They are **the same!** In terms of trees, we represent this by



Homology of $\mathrm{Conf}_n\mathbb{R}^2$

The class in $H_2(\mathrm{Conf}_3(\mathbb{R}^2))$



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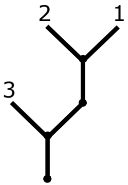

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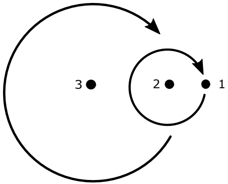

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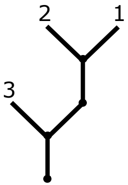

We will write (3(21)) for this class.

Homology of $\mathrm{Conf}_n\mathbb{R}^2$

Using this notation, there is a relation in $H_2(\mathrm{Conf}_3(\mathbb{R}^2))$.

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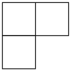

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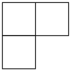

$H_1(\mathrm{Conf}_3\mathbb{R}^2)$ is spanned by $(12) \otimes 3, (23) \otimes 1$ and $(13) \otimes 2$.

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The Lie representations are generated by trees/bracketings, modulo Lie algebra relations

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In general $\mathrm{Lie}(n)$ is an $(n-1)!$ dimensional $\mathbf{S}_n$ representation. There are combinatorial formulas for its character.

Arnold, Cohen computations

Theorem

The $\mathbf{S}_n$ *representation* $H_{nd-1}(\mathrm{Conf}_n(\mathbb{R}^d))$ *is* $\mathrm{sign}^{\otimes d-1}\mathrm{Lie}(n)$*,*

In a sense, these are the "irreducible classes"– the ones which involve all of the points rotating about each other.

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Theorem

Write $\Sigma^d\mathrm{Lie}$ *for* $\bigoplus_n \mathrm{sign}^{\otimes d-1}\mathrm{Lie}(n)[d-1]$*. Write* Com *for the sum of* $\mathbf{S}_n$ *representations* $\bigoplus_n \mathrm{triv}$*. Then we have*

$$\oplus_n H_\bullet(\mathrm{Conf}_n\mathbb{R}^d) = \mathrm{Com} \circ \Sigma^d\mathrm{Lie}.$$

In other words, it is a free non-unital TCA.

This implies a plethystic formula for the symmetric function that records the cohomology of $\mathrm{Conf}_n\mathbb{R}^d$ as an $\mathbf{S}_n$ representation.

The Totaro, Cohen, Kriz model

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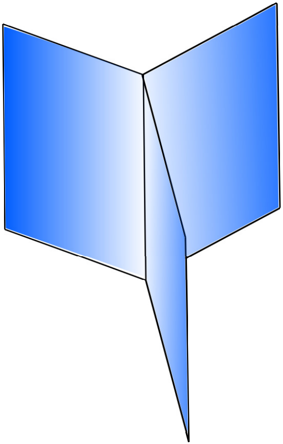
Can use this to compute Euler characteristics / prove representation stability.

Non-Manifolds

What is the local structure of points moving around in $Y \times [0, 1]$?



Algebraic substitute

Let $\mathrm{P}(n)$ be the **poset of set partitions**, ordered by refinement.
For $p \in \mathrm{P}(n)$, we define the closed subset of X^n:

$$Z_p := \{(x_i) \in X^n | x_i = x_j \text{ if } i, j \text{ are in the same block of } p\} = X^{\#p}.$$

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Theorem (T.)

From this chain complex of $\mathrm{P}(n)$ representations, we can build a chain complex that computes $H^*(\mathrm{Conf}_n(X))$.

Spectral Sequence

By **filtering** this construction, we obtain a spectral sequence. The Lie representations appear because they are the homology groups of $\mathrm{P}(n)$.

Theorem (T.)

Let X be any Hausdorff topological space. Let k be a field. There is a spectral sequence

$$\bigoplus_{p+q} E_1^{p,q} = \mathrm{Com}\left(\bigoplus_n H^*(X^n, X^n - \Delta X, k) \otimes \mathrm{sgn}\ \mathrm{Lie}(n)[n-1]\right),$$

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Moduli Spaces of Curves

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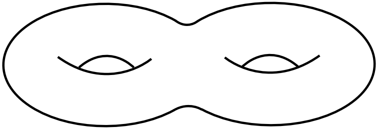
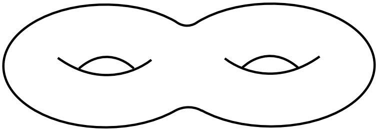
Thus a point in M_2 is a genus 2 curve with a complex structure.

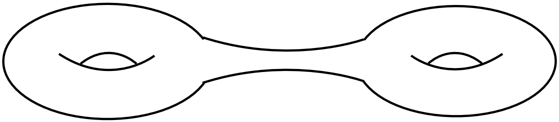


Moduli Space of Curves

It is difficult to visualize complex structures. We can attempt to do it by associating a **hyperbolic metric** to each complex structure (our drawings are not accurate).

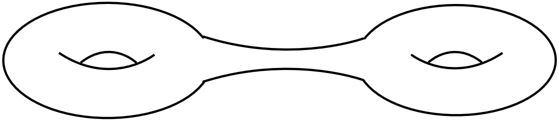


As the complex structure on C changes, the metric deforms, and we trace out a path in M_g.



Compactification by Nodal Curves

As the neck of the surface stretches longer and longer, we obtain a sequence of curves with no limiting smooth curve.



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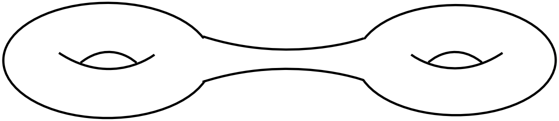
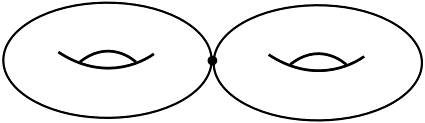


To compactify M_g, we can consider a larger space $\overline{M}_g$ that has nodal curves.



Moduli Spaces of Marked Curves

$M_{g,n}$ is the moduli space of complex curves of genus g and n marked points.

$$M_{g,n} = \frac{\{C \text{ complex genus } g \text{ curve}, p_1, \ldots, p_n \in C\}}{\{\text{ isomorphisms } C \simeq C' \text{ preserving } p_1, \ldots, p_n\}}$$

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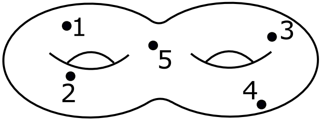
This is a point in $M_{2,5}$.



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The last conditions guarantee that C has finitely many automorphisms that preserve the markings.

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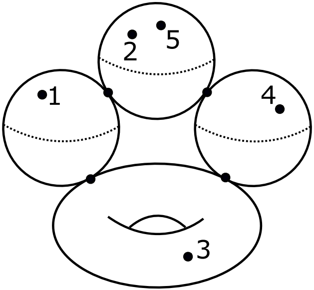
$\overline{M}_{g,n}$ is the **moduli space of stable marked complex curves**.

$$\overline{M}_{g,n} = \frac{\{C, p_1, \ldots, p_n |\ C \text{ is a stable marked curve of genus } g\}}{(C, p_i) \sim (D, q_i) \text{ if } C, D \text{ are isomorphic as marked curves}}.$$

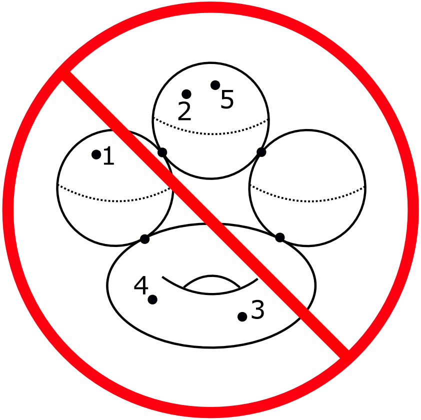
Deligne–Mumford Compactifications

The following nodal curve is stable, and so defines an element of $\overline{M}_{2,5}$



Deligne–Mumford Compactifications

This nodal curve is not stable, because one of the genus 0 components has only 2 special points



Why $M_{g,n}$ and $\overline{M}_{g,n}$?

Here are some reasons to study the homology of $M_{g,n}$ and $\overline{M}_{g,n}$

- Classes in $H^i(M_{g,n})$ are characteristic classes for families of smooth marked curves.
- The compactification $\overline{M}_{g,n}$ has an intersection theory, related to 2-dimensional gravity.
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The homology is well understood for $g \gg n, i$. Little is known about the range $g > i$.

Representation Stability for $M_{g,n}$

Theorem (Jiménez Rolland)

Fix i and g. The cohomology $H^i(M_{g,n}, \mathbb{Q})$ carries the strucutre of a finitely generated **FI** *module.*

In particular, this implies that the function $n \mapsto \dim H^i(M_{g,n})$ agrees with a polynomial for $n \gg 0$.

Homology of $\overline{M}_{0,n}$

The (co)homology of $\overline{M}_{0,n}$ is completely known. It was first computed by Keel.

Theorem (Keel)

The vector space $H^2(\overline{M}_{0,n})$ has dimension $2^{n-1} - \frac{n^2-n+2}{2}$.

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Theorem (Keel)

The vector space $H^2(\overline{M}_{0,n})$ has dimension $2^{n-1} - \frac{n^2-n+2}{2}$.

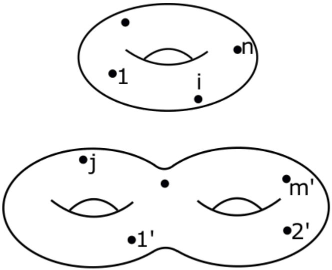
Thus $H^2(\overline{M}_{0,n})$ cannot be a finitely generated **FI** module.

A **different** algebraic structure is required to study $H_i(\overline{M}_{g,n})$.

An algebraic structure: Gluing maps

Let $i \in [n] = \{1, \ldots, n\}$ and $j \in [m] = \{1', \ldots, m'\}$. There is a gluing map

$$\mathrm{glue}_{i,j} : \overline{M}_{g,n} \times \overline{M}_{g,m} \to \overline{M}_{g,n+m-2}$$



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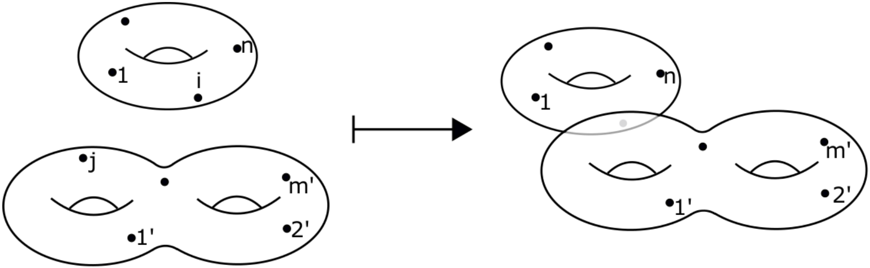

$\mathbf{FS}^{\mathrm{op}}$ modules

FS is the category of finite sets and surjections.

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An $\mathbf{FS}^{\mathrm{op}}$ module is a **contravariant functor** from **FS** to the category of abelian groups.

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$\mathbf{FS}^{\mathrm{op}}$ modules are a different enhancement of sequences of $\mathbf{S}_n$ representations than **FI** modules.

They are less well understood: notice that $\mathbf{FS}(n,k)$ grows exponentially $O(k^n)$, whereas $\mathbf{FI}(k,n)$ only grows polynomially $O(n^k)$.

An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

The category $\mathbf{FS}^{\mathrm{op}}$ is generated by permutations $\sigma \in \mathbf{S}_n$, and surjections $[n+1] \twoheadrightarrow [n]$.

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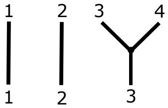
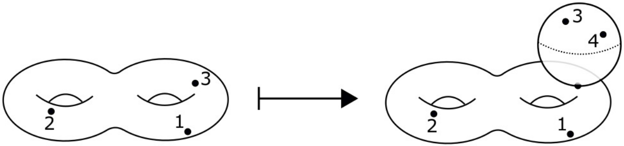


we associate a map $f : \overline{M}_{g,3} \to \overline{M}_{g,4}$:



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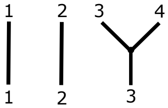
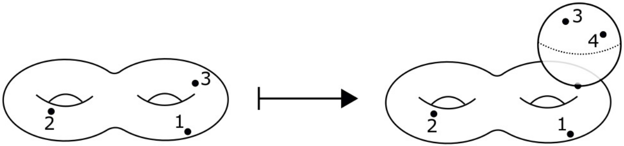
- To the surjection $[4] \twoheadrightarrow [3]$



we associate a map $f : \overline{M}_{g,3} \to \overline{M}_{g,4}$:



On homology, this defines an action of $\mathbf{FS}^{\mathrm{op}}$.

An $\mathbf{FS}^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

Why? At space level this construction generates an action by the category, **BT**, of binary forests.

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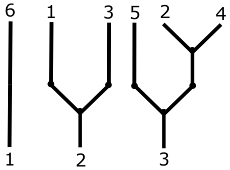

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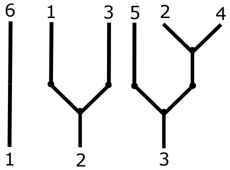

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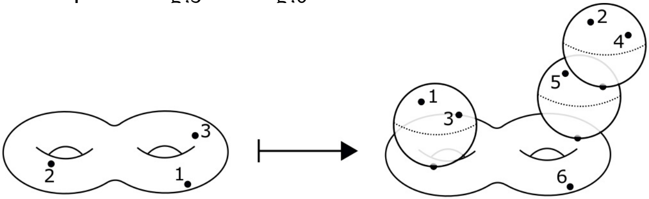
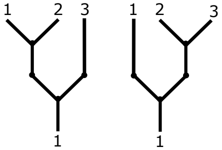


An **FS**$^{\mathrm{op}}$ module structure on $H_i(\overline{M}_{g,n})$

On homology, different trees induce the same map.

For example, the trees



give two maps $\overline{M}_{g,1} \underset{T_2}{\overset{T_1}{\rightrightarrows}} \overline{M}_{g,3}$.

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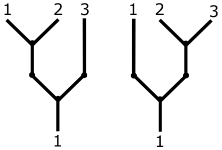

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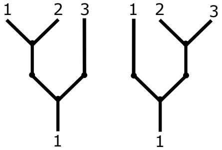

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T_1 and T_2 are given by evaluating this gluing map at two different points of $\overline{M}_{0,4}$. Since $\overline{M}_{0,4}$ is connected, there is a path between them, and they induce the **same** map on homology.

Finite generation

Arguing like this, we see that the action of **BT** induces an $\mathbf{FS}^{\mathrm{op}}$ action on homology.

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Theorem

Let $g, i \in \mathbb{N}$. Then the $\mathbf{FS}^{\mathrm{op}}$ module

$$n \mapsto H_i(\overline{M}_{g,n}, \mathbb{Q})$$

is a subquotient of an $\mathbf{FS}^{\mathrm{op}}$ module that is finitely generated in degree $\leq p(g, i)$ where $p(g, i)$ is a polynomial in g and i of order $O(g^2 i^2)$.

Consequences of finite generation

Applying results of Sam–Snowden on finitely generated $\mathbf{FS}^{\mathrm{op}}$ modules, we obtain the following

Corollary

Let $C = p(g, i)$. Then

- *The generating function for the dimension of $H_i(\overline{M}_{g,n})$ is rational and takes the form*

$$\sum_n \dim H_i(\overline{M}_{g,n}) t^n = \frac{f(t)}{\prod_{j=1}^{C}(1 - jt)^{d_j}}$$

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- *In particular, there exist polynomials $f_1(n), \ldots, f_C(n)$ such that for $n \gg 0$ we have $\dim H_i(\overline{M}_{g,n}) = \sum_{j=1}^{C} f_j(n) j^n$*

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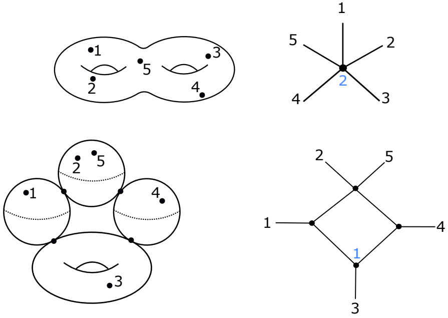
- *Let $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_C$ be an integer partition of k, and $\lambda + n$ be the partition $\lambda_1 + n \geq \lambda_2 \geq \cdots \geq \lambda_C$. The multiplicity of $\lambda + n$ in $H_i(\overline{M}_{g,n+k})$,*

$$n \mapsto \dim \mathrm{Hom}_{\mathbf{S}_{n+k}}(M_{\lambda+n}, H_i(\overline{M}_{g,n+k})),$$

is bounded by a polynomial of degree $C - 1$.

Idea of proof

To each curve, we can associate a stable graph. This stratifies $\overline{M}_{g,n}$



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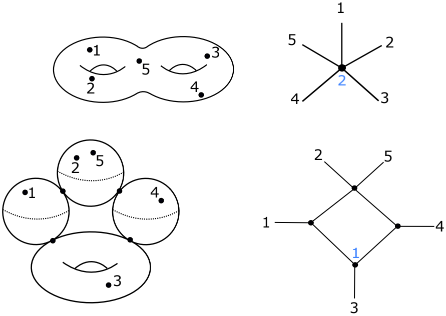
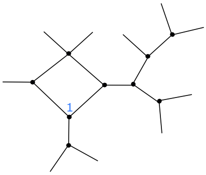


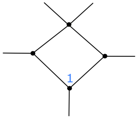
To bound $H_i(\overline{M}_{g,n})$, we bound the (Borel–Moore) homology of these strata.

Idea of proof

BT acts on the strata by tacking on trees. Any homology class coming from this stratum



is pushed forward from a smaller stratum



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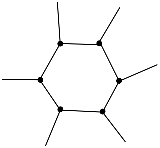

This is a possible exception.

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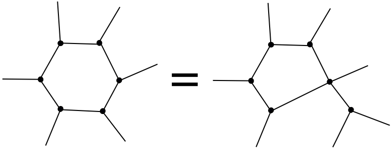

This is a possible exception. But any class from first stratum should be a homologous to one from the second.

Technical problems

The category **BT** is not known to be Noetherian, and $\mathbf{FS}^{\mathrm{op}}$ does not act on the spectral sequence associated to the filtration.

Need to define a **coarsening** of the stratification to make the argument work– requires more combinatorics and some algebraic geometry.

Thanks!

Thanks Andrew!

Thanks to my committee: David, Jenny, Karen, and Venky.

Thanks to my parents for travelling here from NH!

Thanks Alyssa!