Cutting and Pasting

rethinking how we measure

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UMich Undergrad Math Club, March 2018

Rethinking size

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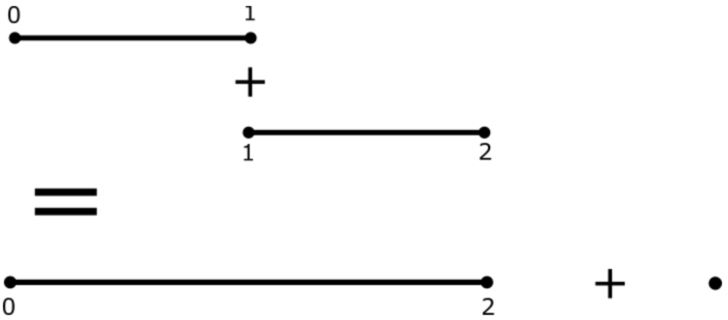
What happens when we measure shapes while taking this extra point into account?

Which is a better meter stick?

If $[0,1]$ and $[0,1)$ no longer have the same size, then we have to choose which one to measure length with.

Closed intervals = Bad

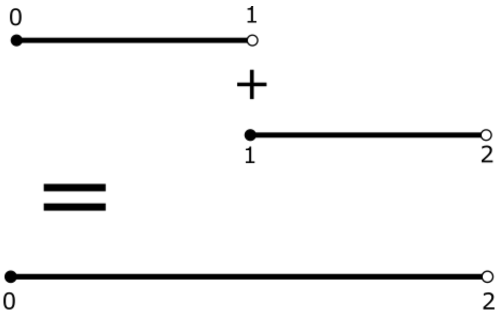
If we use $[0, 1]$ as our unit of size $1\mathbf{m}$, then two copies of $[0, 1]$ would have size $2\mathbf{m}$.



But $[0, 2]$ should also have length $2\mathbf{m}$, and there is a point left over!

Half-open intervals = Good

If we use the half interval, there is no overlap, so things work out better.



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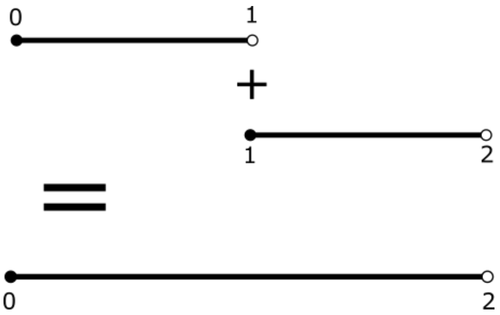

We will use half open intervals as our meter sticks!

Size

We will measure shapes in metric units $\mathbf{m}$ = meters.

For example:

- $[0, 1)$ has size $1\mathbf{m}$,
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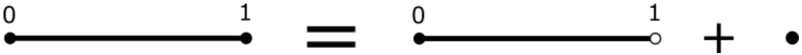

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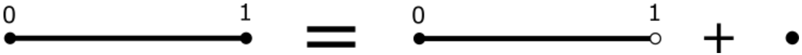

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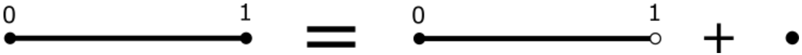
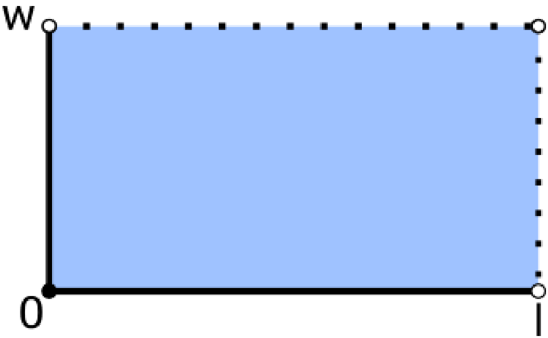


Similarly $(0, 1)$ has size $1\mathbf{m} - 1$.

What happens when we measure other shapes using our new convention?

Rectangles

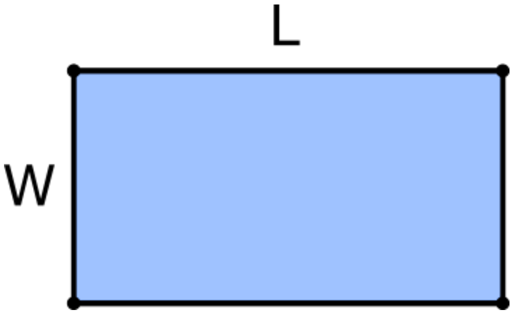


The size of the rectangle $[0, l) \times [0, w)$ is

$$l\mathbf{m} \cdot w\mathbf{m} = lw\ \mathbf{m}^2.$$

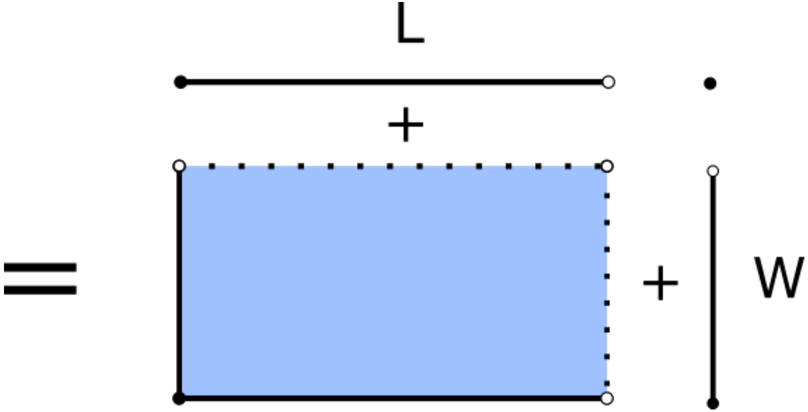
Rectangles

What about a closed rectangle?



Rectangles

We break it up into pieces



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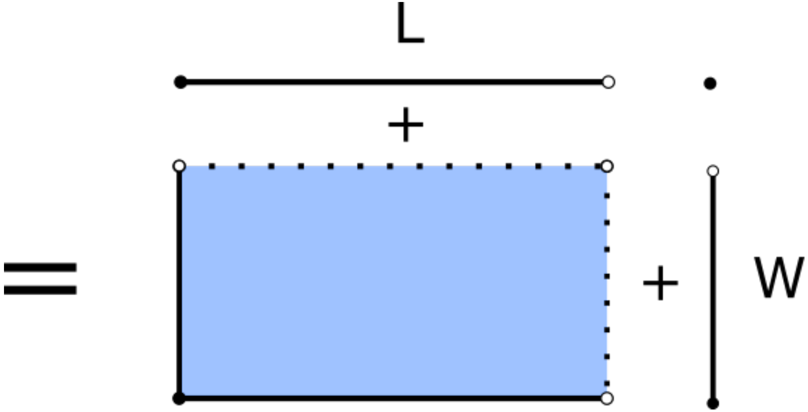
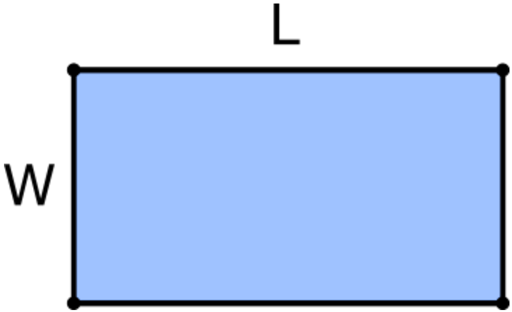


Collecting terms, we get $LW\mathbf{m}^2 + (L+W)\mathbf{m} + 1$.

Rectangles



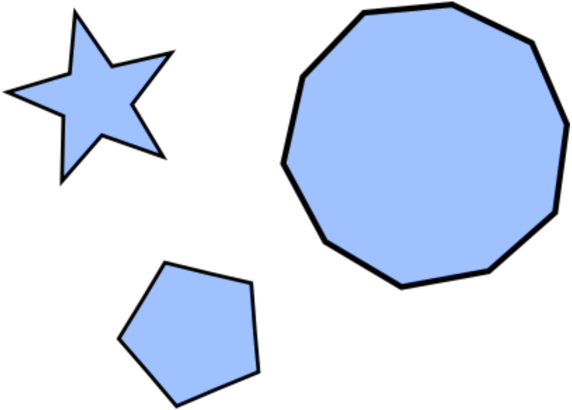
We also could have calculated

$$(L\mathbf{m}+1)(W\mathbf{m}+1) = LW\mathbf{m}^2 + (L+W)\mathbf{m} + 1$$

and gotten the same answer!

Other shapes?

Rectangles are somewhat plain shapes. What are the sizes of these?



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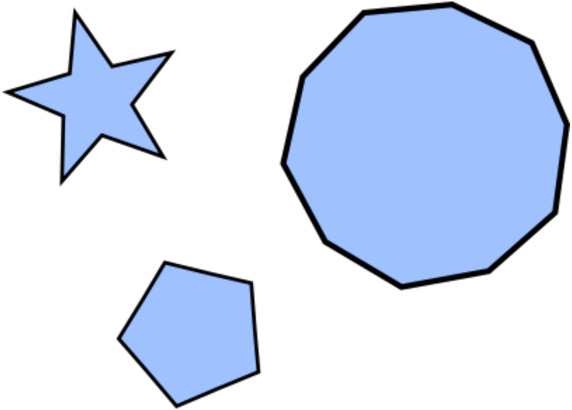
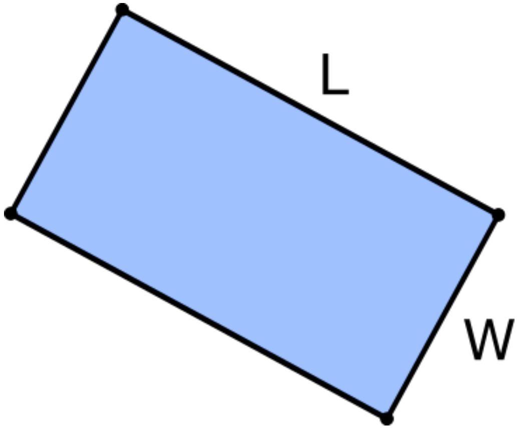


To find out, we will have to go through several steps

Rotations

Tilted rectangles are a bit more exciting than ordinary rectangles



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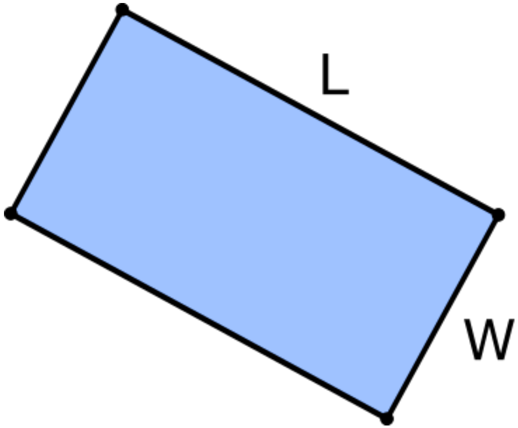

This still has size $LW\mathbf{m}^2 + (L + W)\mathbf{m} + 1$.
(Size is not changed by rotations).

Parallelogram

What is the size of this parallelogram?

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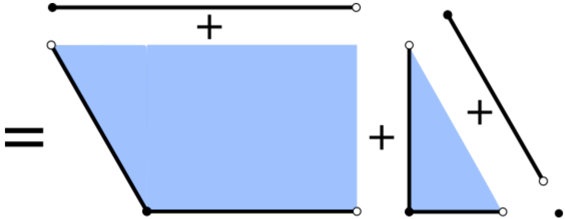
We can find out by breaking it into pieces



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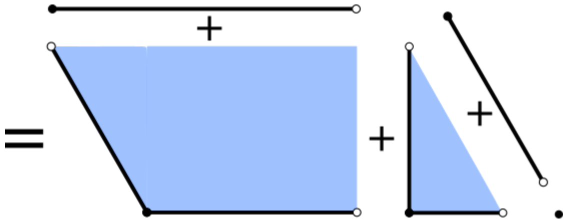
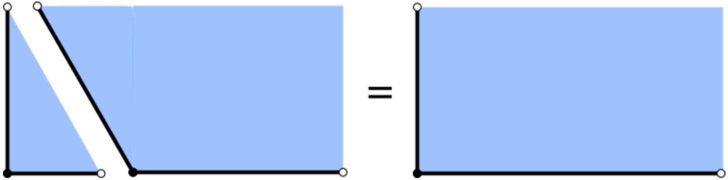


We can rearrange these pieces using the identity



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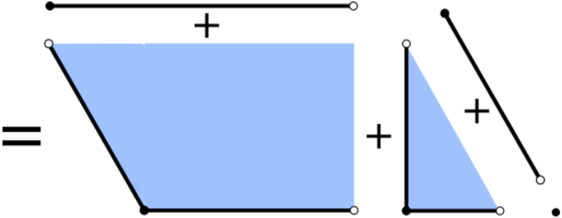


So we have

Parallelogram

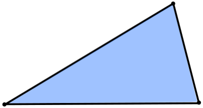
The size of the parallelogram is

$$A\ \mathbf{m}^2 + \frac{1}{2} P\ \mathbf{m} + 1,$$

where A is the area and P is the perimeter.

Triangle

What about a triangle?



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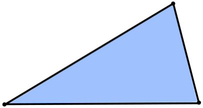
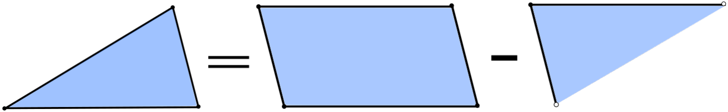


We “double” it to a parallelogram

Triangle

We have



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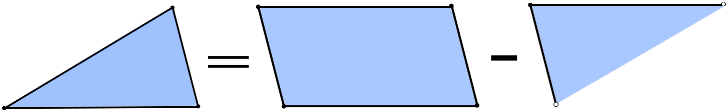
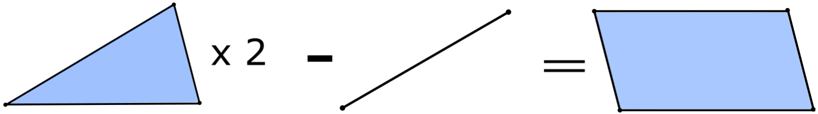


Rearranging, we get



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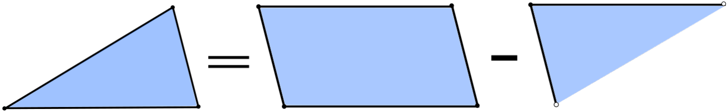

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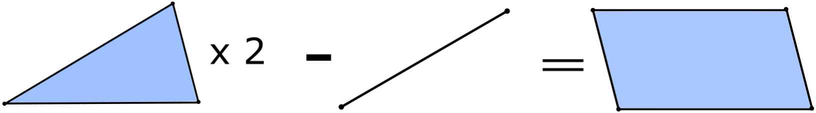

We can solve for the triangle.

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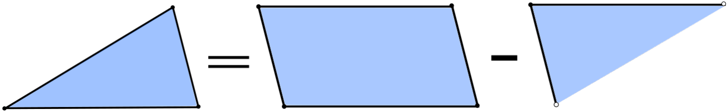

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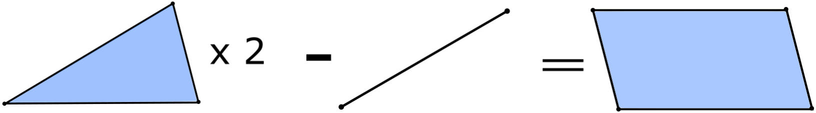
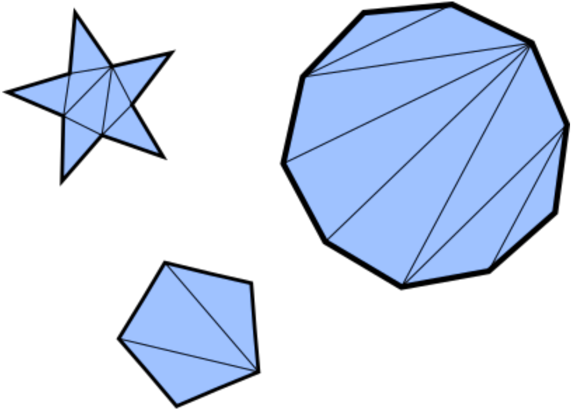


We can solve for the triangle.

$$A\,\mathbf{m}^2 + \frac{1}{2}P\,\mathbf{m} + 1,$$

where A is the area of the triangle and P is the perimeter.

Other shapes, revisited



Now we can calculate the sizes of other shapes by cutting them into triangles.

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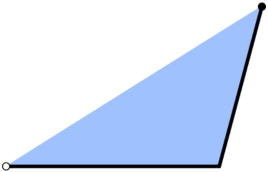
We can prove that this happens for all polygons, by induction on the number of triangles it takes to make them.

Not quite solid shapes

If we subtract half edge from the triangle,



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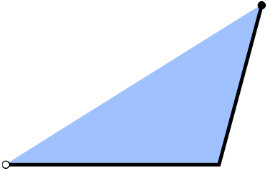

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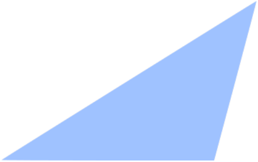

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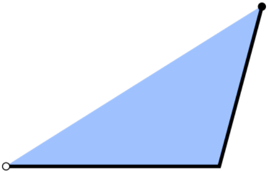

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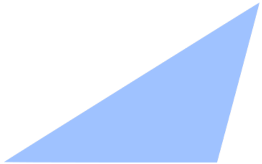

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We might say that closed sides count for $+1/2$ and open sides count for $-1/2$.

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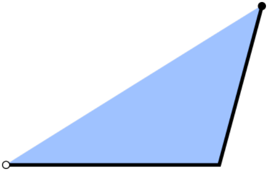

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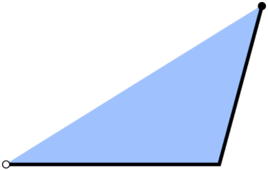

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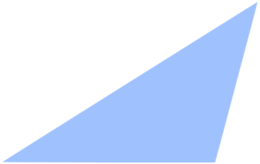

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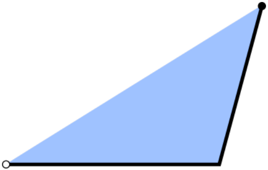

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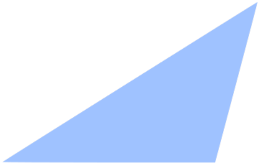

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Closed sides count for $+1/2$ and open sides count for $-1/2$.

One?

The $\mathbf{m}^2$ term is area, and the $\mathbf{m}^1$ term is a weighted count of the perimeter. What is the constant term?

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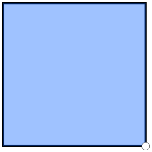

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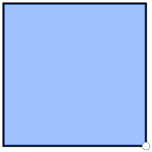

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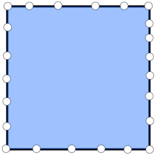

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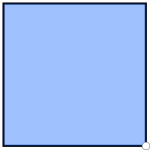

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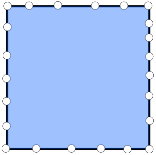

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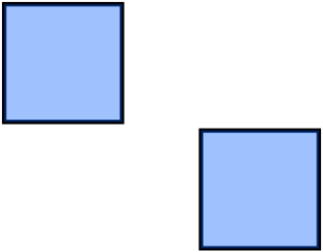
Let's restrict to closed shapes.

Constant term is number

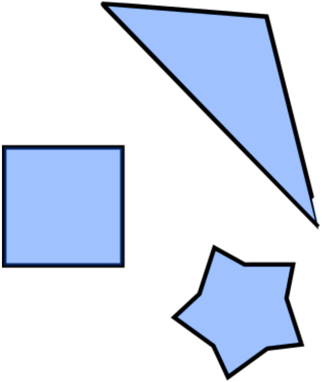
For two squares



we have $2(A\mathbf{m}^2 + 1/2P\mathbf{m} + 1\mathbf{m}^0)$, so the constant term is $2\mathbf{m}^0$.

Constant term is number

For any three shapes the constant term is $3\mathbf{m}^0$.



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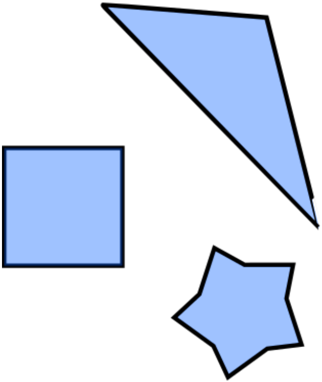

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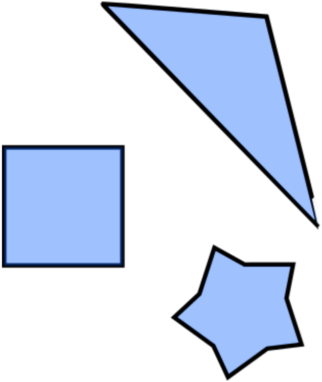
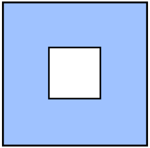


Zero-dimensional measurement is counting!

It doesn't depend on sizes, or whether the shape is a square or a triangle.

Or is it?

What is the size of a square with missing center?



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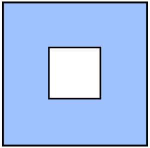
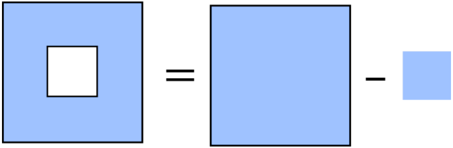


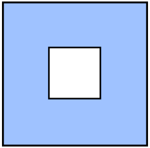
Write S for the big side length, and s for the small one. We have



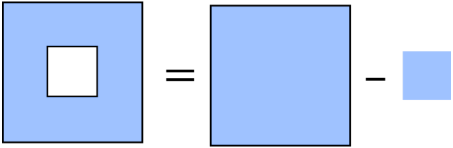
$$(S\mathbf{m}+1)^2-(s\mathbf{m}-1)^2=(S^2-s^2)\mathbf{m}^2+(2S+2s)\mathbf{m}+0.$$

Or is it?

What is the size of a square with missing center?



Write S for the big side length, and s for the small one. We have

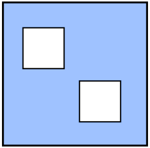


$$(S\mathbf{m}+1)^2-(s\mathbf{m}-1)^2=(S^2-s^2)\mathbf{m}^2+(2S+2s)\mathbf{m}+0.$$

The $\mathbf{m}^2$ and $\mathbf{m}^1$ term are as expected, but the $\mathbf{m}^0$ is zero!

Holes

Each time we subtract another square, we subtract another $(s\mathbf{m}-1)^2$:



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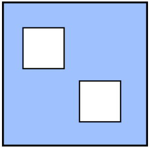

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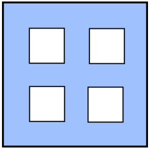
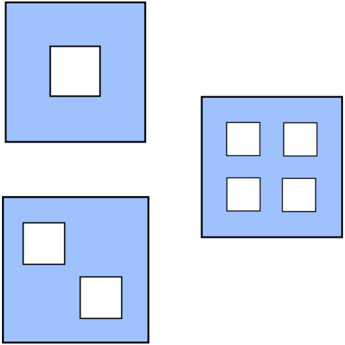


has constant term $-3\mathbf{m}^0$.

Holes

In general, we should add up the number of shapes, and subtract the number of holes.



has constant term

$$3\mathbf{m}^0 - 7\mathbf{m}^0 = -4\mathbf{m}^0.$$

Constant term is **topological** number

We will call the constant term the **number** of a shape.

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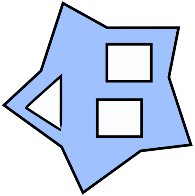
Like before, the number does not care about sizes, or whether the shape is a square or not. For example:



Has number -2.

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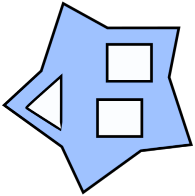

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The number only depends on the **topology** of the shape.

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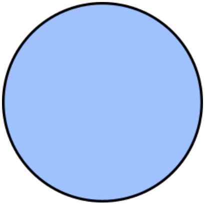
This means we can stretch or deform the shape, and its number stays the same.



The number of the circle is 1,
just like a square or a triangle.

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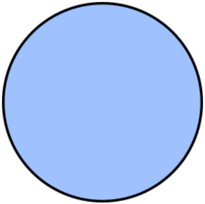

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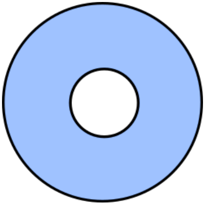
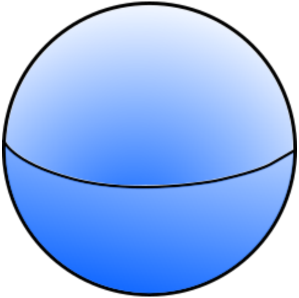


The number of the ring is zero, just like a square with a hole.

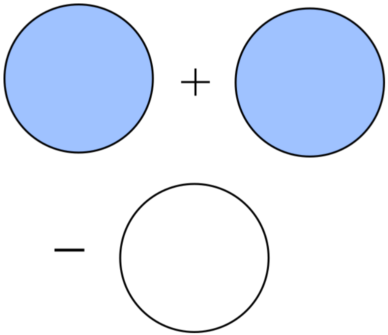
What is the number of a sphere?



Let's not limit ourselves to the plane!

What is the number of a sphere?

We can build the sphere from the top and the bottom



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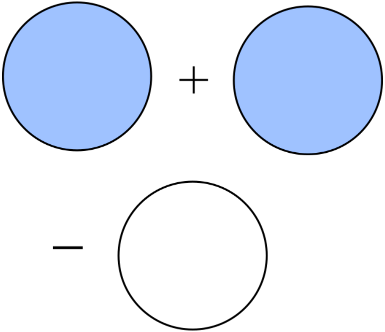

So the number is $1 + 1 - 0 = 2$.

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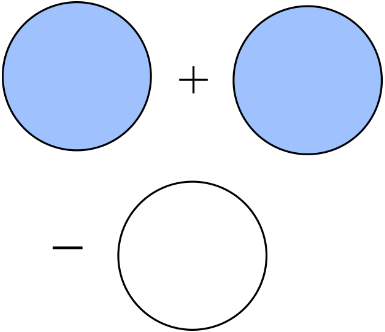
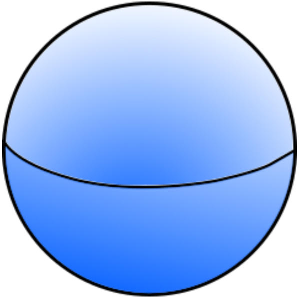


So the number is $1 + 1 - 0 = 2$.

Warning: because we have to stretch the two caps, we can't figure out the $\mathbf{m}^2$ or $\mathbf{m}^1$ term this way.

What is the number of a sphere?

There are many other ways of figuring out the number of a sphere

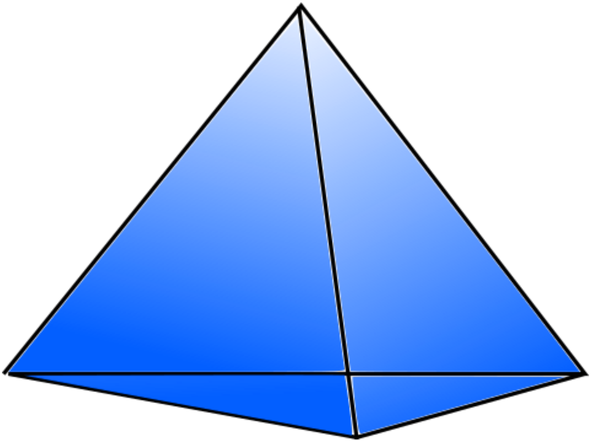


We could break it up as a point plus an open disk, or as a two solid disks plus an open cylinder, etc.

No matter what way you try, you will get the same answer.

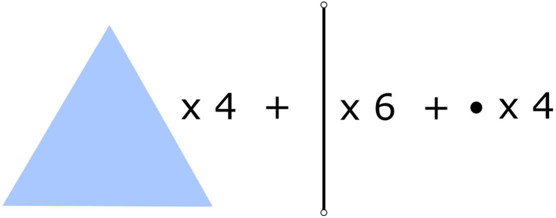
Tetrahedron

What is the size of a (hollow) tetrahedron?



Tetrahedron

Break it into faces, edges, and vertices.



We have

$$4 \cdot (\sqrt{3}/4\mathbf{m}^2 + -3/3\mathbf{m} + 1) + 6 \cdot (1\mathbf{m} - 1) + 4 \cdot 1$$

$$= \sqrt{3}\mathbf{m}^2 - 8\mathbf{m} + 2$$

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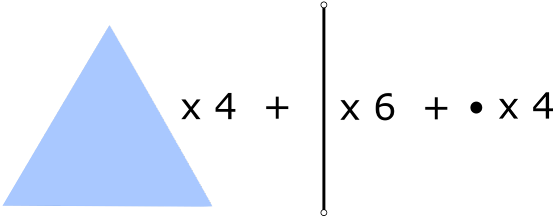

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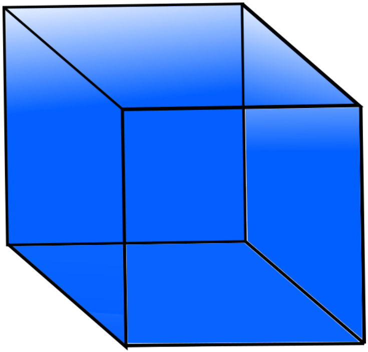
The 2 is because, topologically, the tetrahedron is the same as the sphere!

Cube

We can do the same thing for the cube



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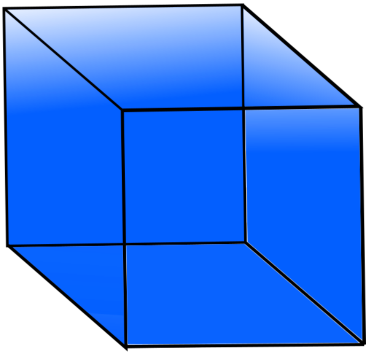
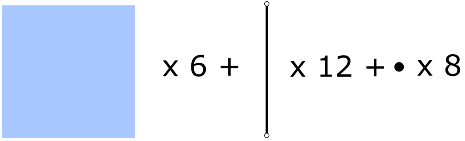


We already know that the constant term will be $2\mathbf{m}^0$, the topological number of the sphere.

Cube



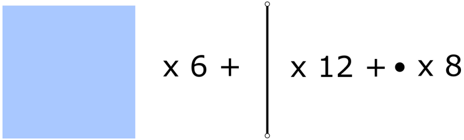
The constant term is

$$F - E + V,$$

the number of faces minus the number of edges, plus the number of vertices. We get

$$6 - 12 + 8 = 2$$

Cube



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This works the same for other polyhedra (octahedra, dodecahedra, soccer balls!).

Euler number

Theorem

For any polyhedron

$$F - E + V = 2,$$

where F, E *and* V *are the number of faces, edges, and vertices respectively.*

Euler number

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where F, E and V are the number of faces, edges, and vertices respectively.

This theorem was first proved by Euler, and the topological number is named after him.

$$\text{topological number of } X = \chi_c(X) = \text{ Euler-characteristic of } X$$

Axioms

Here are the rules that we've used to calculate sizes

- $[0, 1)$ has size $1\mathbf{m}^1$ and $\{1\}$ has size $1\mathbf{m}^0$
- Size is preserved by cutting and pasting
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- The size of $A \times B$ is the size of A times the size of B

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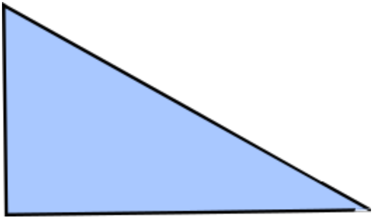
What if we wanted to compute the size of a solid tetrahedron? In higher dimensions, it is harder to cut and paste.

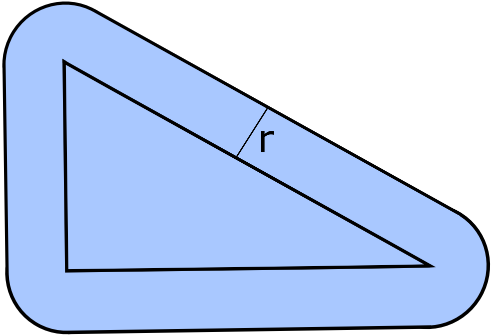
Steiner's formula

Start with a triangle:



Steiner's formula

And consider the set of all points that are within distance r of the triangle:



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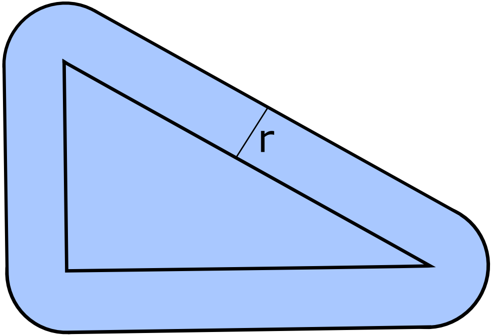
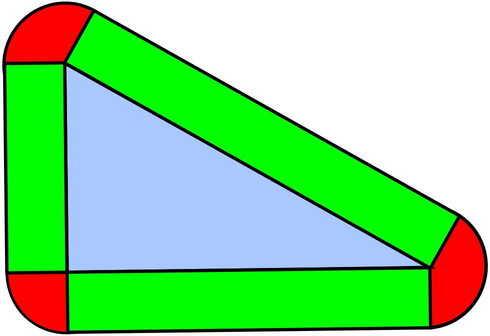


What is the area of this shape as a function of r? (Just plain area!)

Steiner's formula

To find the area, it helps to break the shape into pieces



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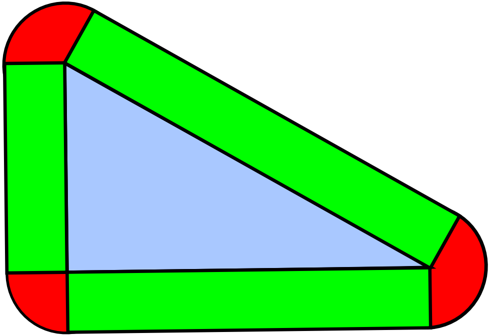
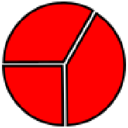


The red pieces fit together!



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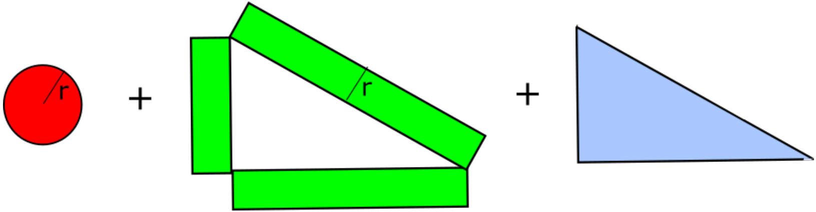
So the total area is



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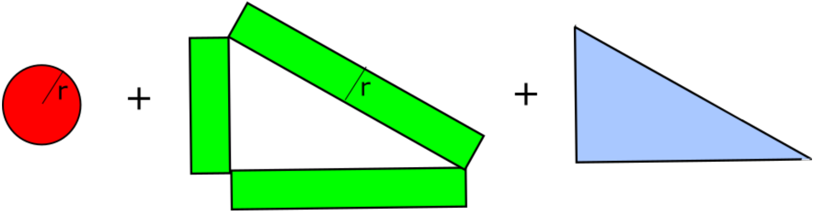
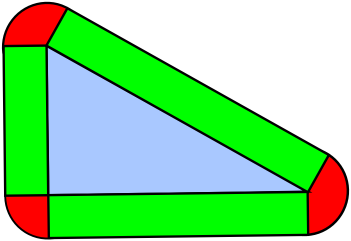


$$= (\pi r^2 + Pr + A)\ \mathbf{m}^2,$$

where P and A are the perimeter and area of the triangle.

Steiner's Formula

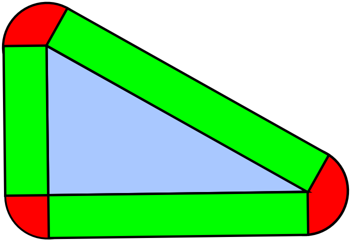


$$(\pi r^2 + Pr + A)\ \mathbf{m}^2$$

Let's rearrange this formula more suggestively.

$$A\ \mathbf{m}^2 \cdot 1 + \frac{P}{2}\ \mathbf{m} \cdot 2r\mathbf{m} + 1\mathbf{m}^0 \cdot \pi r^2 \mathbf{m}^2$$

Steiner's Formula



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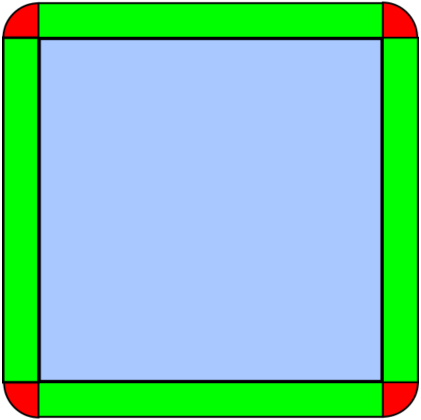
The coefficients $1, 2r\mathbf{m}, \pi r^2 \mathbf{m}^2$ are the $0, 1, 2$ volume of the $0, 1, 2$ disk.

$$D_d(r) = \{(x_i)_{i=1}^d \in \mathbb{R}^d \mid \sum_i x_i^2 \leq r\}$$

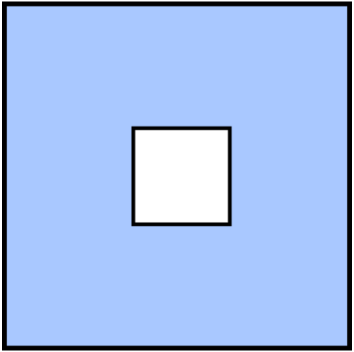
Steiner's formula

The same formula works for other polygons



Counterexample

But not for shapes that are not convex



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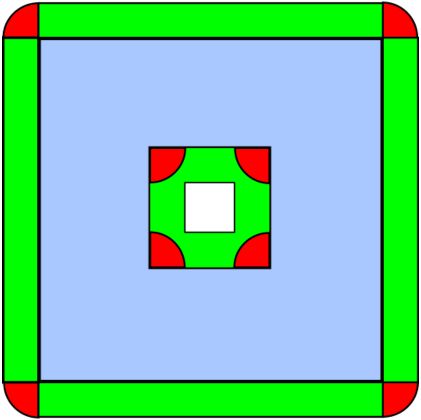

Steiner's formula

Theorem (Steiner)

For any convex body, $X \subset \mathbb{R}^d$, the volume of the set of points within distance r of X is

$$\mathrm{vol}_d(X) + \mathrm{vol}_{d-1}(X) \cdot 2r + \cdots + \mathrm{vol}_{d-i}(X) \cdot k_i r^i \cdots + 1 \cdot k_d r^d$$

where k_i is the volume of D_i, and $\mathrm{vol}_i(X)$ is independent of r.

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The numbers $\mathrm{vol}_i(X)$ are exactly what we've been measuring!

$\mathrm{vol}_i(X)$ is called the ***i*th intrinsic volume** of X.

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So the size of the unit ball is

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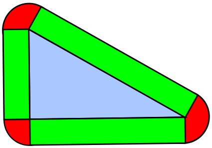
What is the length of a ball? **Four**!

Angles

Steiner's formula suggests that **angles** are related to topological number.



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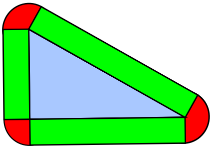

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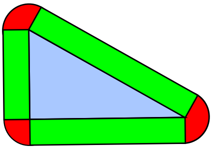

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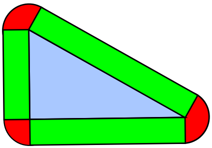

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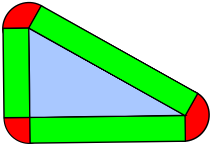

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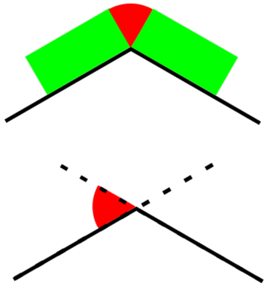
Let's examine this more

Angles

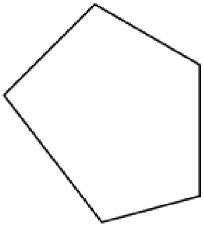
It helps to think of the angle in Steiner's formula as the external angle



The red angles in both pictures are the same.

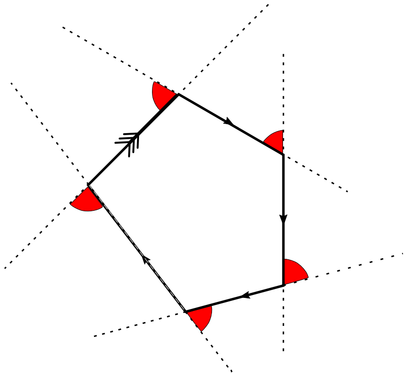
Angles

The unfilled pentagon has topological number 0.



Angles

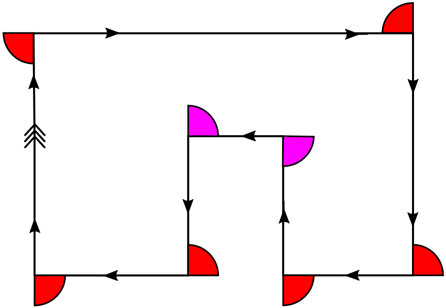
We can think of the fact that the Steiner angles add up to one in terms of an ant walking around the path.



The angles add up to 2π because the ant turns around once.

Angles

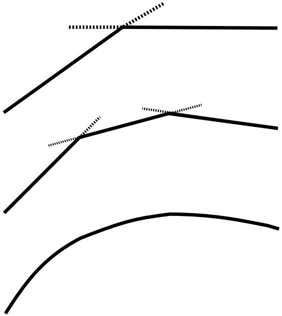
This also works for paths that are not the boundary of a convex shape.



The pink angles cancel two of the red angles, so we get 2π in total.

From angles to curvature

If we want to follow paths that are not just straight lines, then we can take the limit of straight approximations



For a curve $(x(t), y(t))$, the infinitesimal "external angle" is given by the curvature

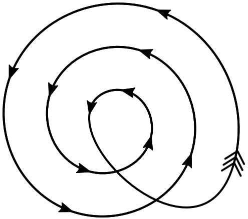
$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

Integrating curvature

The integral

$$K = \int k(t)\mathrm{d}t$$

is the total curvature of the curve (just like the sum of the angles).



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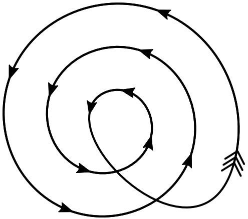

Theorem (Gauss)

The integral K is always $2\pi N$ where $N - 1$ is the number of times the curve intersects itself (counted with multiplicity).

This is just the beginning of the interaction between curvature and topology!

Thanks for coming!

For more, see *What is the length of a potato?* by Steven Schanuel

Some related key words:

- Quermassintegrals
- Weyl's tube theorem
- Curvature measures
- Gauss Bonnet
- Polytope algebra