Inclusion Exclusion and Rep Stability for Configurations in Non-Manifolds

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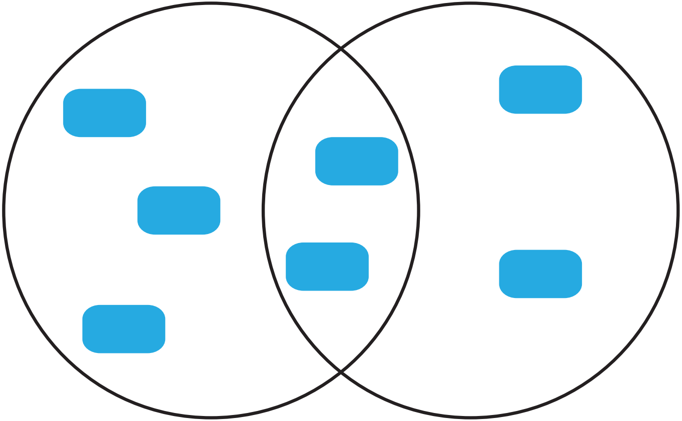
University of Michigan

Young Topologists Meeting, July 2017

Inclusion-Exclusion

How many blue blobs are there?



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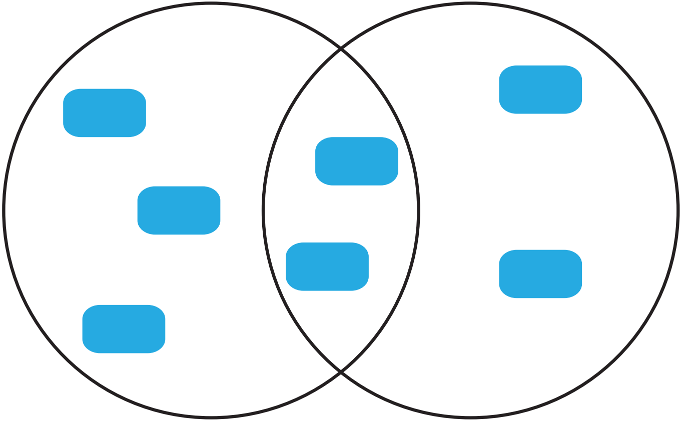
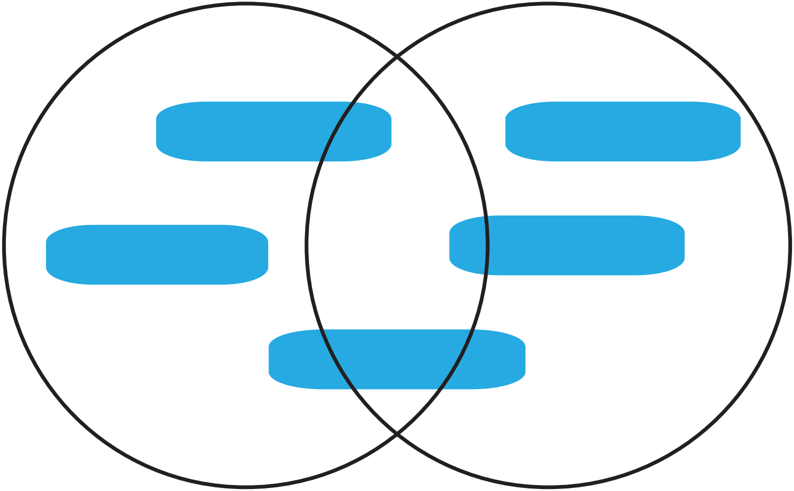


There are $5 + 4 - 2$, so 7 in total

With Overlaps

Does inclusion exclusion still work when there are overlaps?



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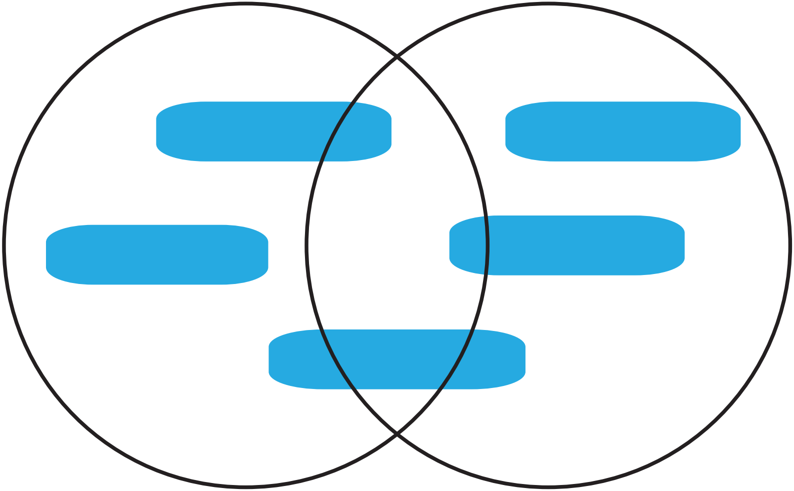
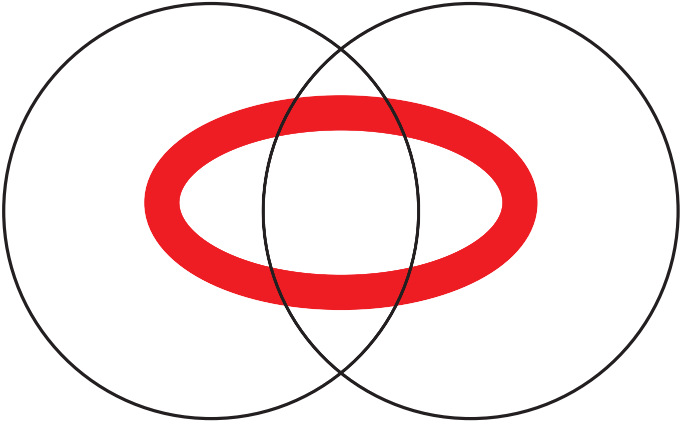


Yes! We have $4 + 4 - 3 = \color{red}{5}$

Or does it?

We have $1 + 1 - 2 = 0$. What is going on?



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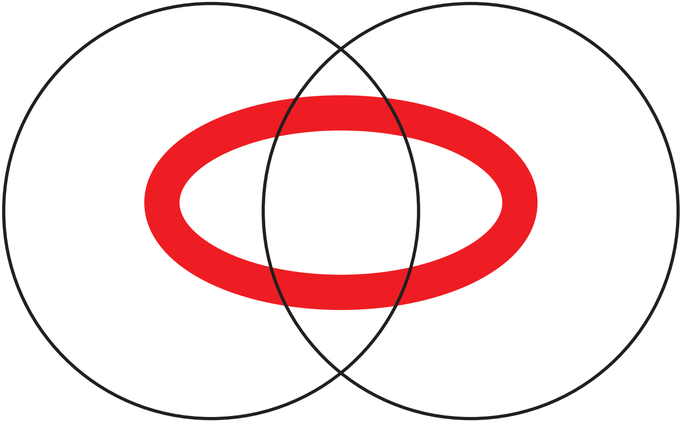
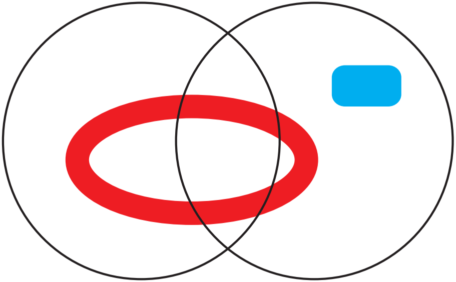


0 is the **Euler Characteristic** of the annulus.



Theorem

For A and B open subsets of X we have

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$$

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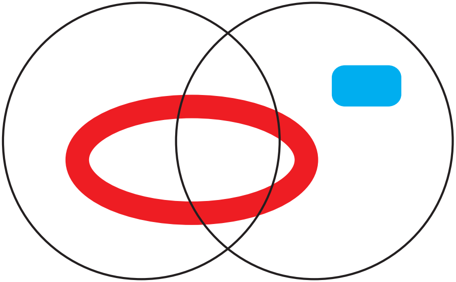

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Proof.

Mayer-Vietoris:

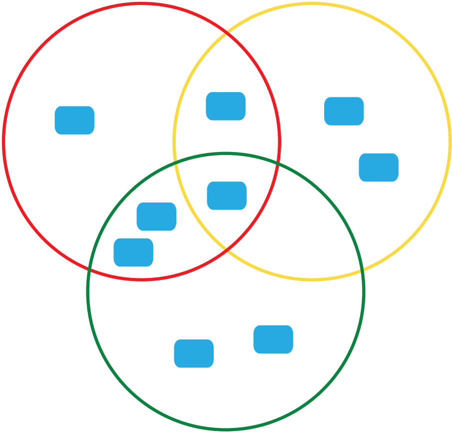
$$H^*(A \cup B) \to H^*(A) \oplus H^*(B) \to H^*(A \cap B)$$

+**Fundamental fact**: If C bounded complex then

$$\sum_i (-1)^i [H^i(C)] = \sum_i (-1)^i [C^i]$$

□

3-way Inclusion Exclusion



$$\chi(A)+\chi(B)+\chi(C)-\chi(A\cap B)$$

$$-\chi(A\cap C)-\chi(B\cap C)+\chi(A\cap B\cap C)$$

$$=\chi(A\cup B\cup C)$$

n-way Inclusion Exclusion

For A_i open subsets of X we have

$$\chi(\bigcup_{i=1^n} A_i) = \sum_{I \subset [n]} \chi(\cap_{i \in I} A_i)(-1)^{|I|}$$

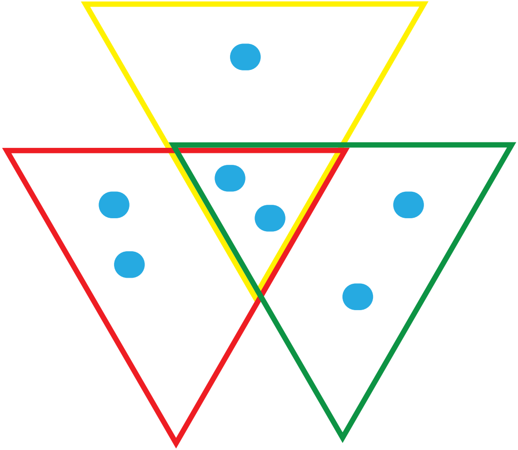
Theorem (Mayer-Vietoris Spectral sequence)

There is a convergent spectral sequence $E_2^{p,q} \implies H^{p+q}(\bigcup_{i \in [n]} A_i)$ with

$$E_2^{p,q} = \bigoplus_{I \subset [n],\ |I| = q} H^p(\cap_{i \in I}\ A_i)$$

Degenerate Inclusion-Exclusion

If the intersections in the Venn diagram are redundant, then terms in the inclusion exclusion formula cancel:



$$\chi(A \cup B \cup C) = \chi(A) + \chi(B) + \chi(C) - 2\chi(A \cap B \cap C)$$

Mobius Numbers

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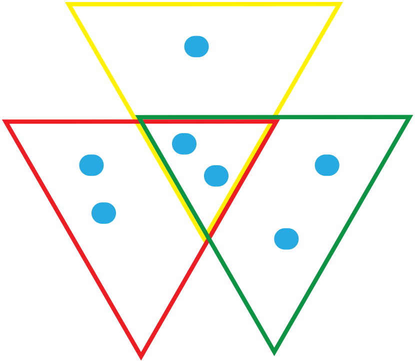
- Equivalently, $\mu(U)$ is a homological invariant of L

$$\mu(U) = \sum_i (-1)^i \dim \mathrm{Ext}^i_L(\mathbb{Z}, \mathbb{Z}_U)$$

Degenerate Inclusion Exclusion



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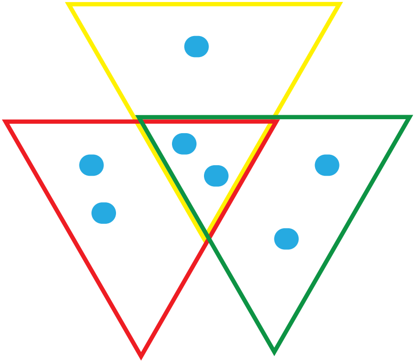

- We can refine $\mu(U)$ to "mobius betti numbers"

$$\mu^i(U) = \dim \widetilde{H}^{i-1}(N(L_{<U}))$$

- And we can incorporate these cohomology groups into a spectral sequence

Theorem (Mobius Mayer-Vietoris)

There is a spectral sequence $E_2^{p,q} \implies H^{p+q}(\bigcup_{U \in L} U)$ *with*

$$\bigoplus E_2^{p,q} = \bigoplus_{U \in L} \tilde{H}^{p-1}(N(L_{<U}), H^q(U))$$

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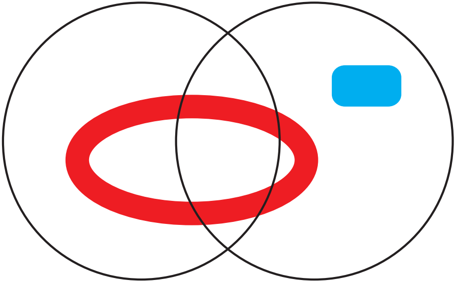

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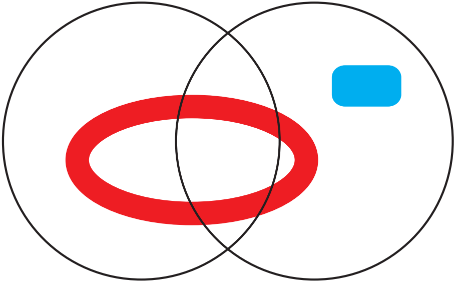

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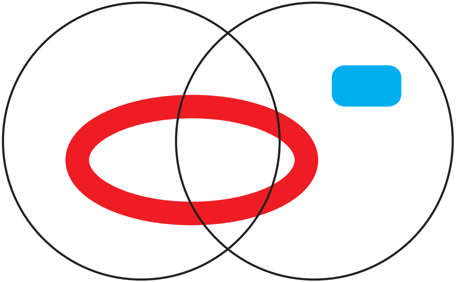
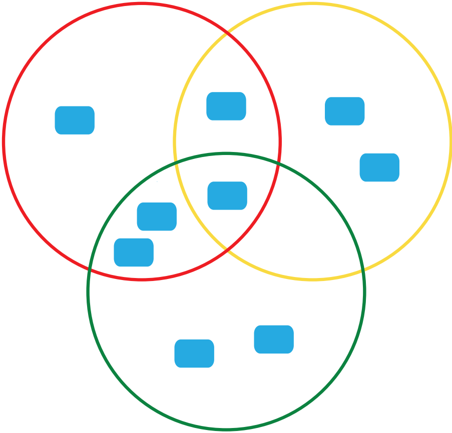


$$1 + 2 - 1 = 2$$

Backwards inclusion exclusion $n = 3$



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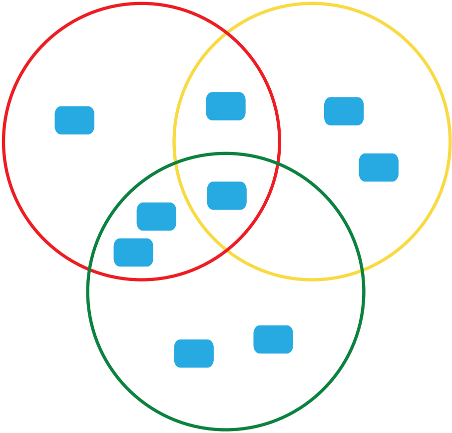

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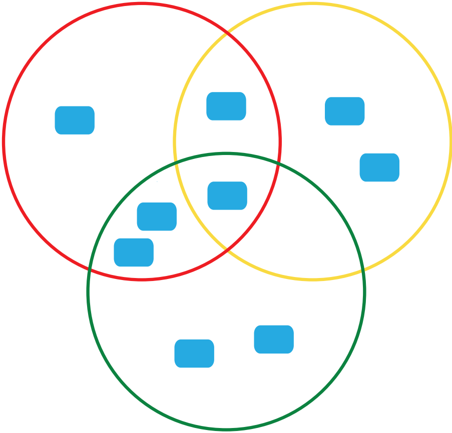

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$$= 1$$

$$= \chi(A \cap B \cap C)$$

Backwards Mayer-Vietoris

- It's easy to prove that backwards inclusion-exclusion works for sets: just take complements!
- But this doesn't "categorify" in an elementary way

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Let Q be the poset formed by the finite **unions** of the U_i ordered by reverse inclusion. It is now a lattice under unions.

Theorem (T.)

There is a spectral sequence $E_1^{-p,q} \implies H^{-p+q}(\cap_i U_i)$ with

$$\bigoplus_{p,q} E_1^{-p,q} = \bigoplus_{p,q} \bigoplus_{U \in Q} \tilde{H}_{p-1}(N(Q_{>U}), H^q(U))$$

This is backwards Mayer-Vietoris (with mobius coefficients).

Configuration Space

- X is a Hausdorff topological space
- The **ordered configuration space** of n points in X is

$$\mathrm{Conf}_n(X) = \{(x_i) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

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- The lattice generated by U_{ij} under intersections is the lattice of set partitions

Question:

What is the (co)homology of $\mathrm{Conf}_n(X)$? As an $\mathbf{S_n}$ representation?

- Answer should involve the cohomology of X
- But $X \mapsto \mathrm{Conf}_n(X)$ does not preserve homotopy equivalences– this suggests we need more!
- When $n >> 0$, we hope that $H^i(\mathrm{Conf}_n(X))$ admits a uniform description.
- For manifolds, the work of Totaro/Cohen gives the main tool to answer this question, using knowledge about configurations in $\mathbb{R}^n$
- Using this work, Church established that the cohomology of configurations in $Conf_n(M)$ satisfies rep stability.

A Variant of Backwards Mayer-Vietoris

There is a more useful version that uses $H^i(X, U) = H^i_Z(X)$ where $Z = U^c$ instead of U.

- By excision, these groups only depend on a neighborhood of Z in X.
- When Z is an orientable submanifold, $H^i_Z(X) = H^{i-\dim Z}(X)$.

Theorem (T.)

There is a spectral sequence $E_1^{p,q} \implies H^{p+q}(\cap_i U_i)$ with

$$\bigoplus_{p,q} E_1^{p,q} = \bigoplus_{U \in L'} \tilde{H}_{p-2}(N((\hat{1}, U)), H^q(X, U))$$

This extends the tools of Totaro/Cohen to non-manifolds.

Rep Stability for Non Manifolds

We say that a point $p \in X$ is a **roadblock** if for $U \ni p$ a contractible neighborhood, $U - p$ is disconnected.

- Every point of a graph G is a roadblock
- $G \times \mathbb{R}^1$ has no roadblock points
- A wedge of two spaces has a roadblock point at the wedge
- Two spheres glued along an edge have no roadblocks

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Let X be a finite connected CW complex. If X has **no roadblocks** *then*

$$n \mapsto H^i(\mathrm{Conf}_n(X), \mathbb{Q})$$

is **representation stable***: in particular its dimension eventually agrees with a polynomial in n.*

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Extends to reasonable connected locally contractible closed subsets of $\mathbb{R}^n$.

For fields $k \neq \mathbb{Q}$, the same criterion implies finite generation of FI modules

Other Directions, Other Arrangement Complements?

Given $U_i \subset X$ with $Z_i = X - U_i$, intersecting the U_i is the same as removing the union of the Z_i

$$\bigcap_i U_i = X - \bigcup_i Z_i$$

The spectral sequence from before gives a tool for computing the cohomology from $H^*_{Z_i}(X)$ and poset homology.

When X is a vector space and the Z_i are subspaces, it degenerates at E_1 and we recover a formula for the cohomology due to Goresky and MacPherson.

Acknowledgements

Thank you for coming!

For More Info I

P. Tosteson
Lattice Spectral Sequences and Cohomology of Configuration Spaces.
https://arxiv.org/abs/1612.06034